A harder quadratic problem

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Question

Let $y = \dfrac{x^2 - 1}{x^2 - (3+k)x + 3k}$

(a) Find a necessary and sufficient condition on k if y can take all real values for all real x.

(b) If $k = 3$, find the least integral value that y can take.

Solution

$$y = \frac{x^2 - 1}{x^2 - (3+k)x + 3k} \Rightarrow y\left[x^2 - (3+k)x + 3k\right] = x^2 - 1$$

$\therefore \quad (y - 1)x^2 - (3 + k)yx + (3ky + 1) = 0$ …. **(1)**

(a) **(1)** is a quadratic if the coefficient of x^2 term is non-zero. If this coefficient is zero, it is a linear equation. Therefore we need to separate into two cases :

(i) $y = 1 \quad \Leftrightarrow \quad (3 + k)yx = 3k + 1 \quad \Leftrightarrow \quad x = \dfrac{3k+1}{3+k}$ …. **(2)**

We get only a point $\left(\dfrac{3k+1}{3+k}, 0\right)$ if $k \neq 3$.

(ii) $y \neq 1$,

For real x, Δ_x of $(*) \geq 0$.

$[-(3+k)y]^2 - 4(y-1)(3ky+1) \geq 0$

$k^2y^2 + 6ky^2 + 9y^2 - 12ky^2 + 12ky - 4y + 4 \geq 0$

$(k - 3)^2 y^2 + 4(3k - 1)y + 4 \geq 0$ …. **(3)**

In order that y can take all real values, then from (1), $k \neq 3$.

Now consider the LHS of (2) as quadratic expression in y, $g(y) = (k - 3)^2 y^2 + 4(3k - 1)y + 4$

$\Leftrightarrow \quad k \neq 3 \quad \wedge \quad \Delta_y = 16(3k - 1)^2 - 4(k - 3)^2(4) \leq 0$

$\Leftrightarrow \quad k \neq 3 \quad \wedge \quad (3k - 1)^2 - (k - 3)^2 \leq 0$

$\Leftrightarrow \quad k \neq 3 \quad \wedge \quad (k + 1)(k - 1) \leq 0$

$\Leftrightarrow \quad k \neq 3 \quad \wedge \quad -1 \leq k \leq 1$

$\Leftrightarrow \quad -1 \leq k \leq 1$

Hence y can take all real values if and only if $-1 \leq k \leq 1$.

(b) If $k = 3$, then $y = \dfrac{x^2 - 1}{x^2 - 6x + 9}$

Using the result in **(a)**,

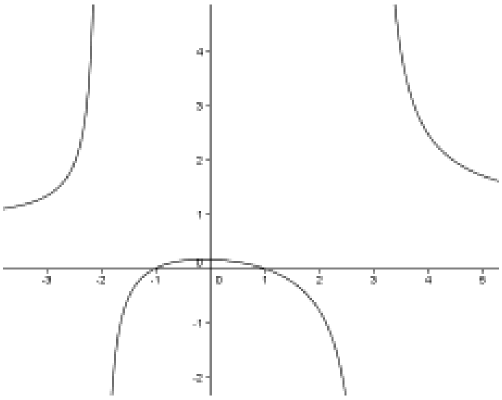
(i) If $y = 1$, then from (2), $x = \dfrac{3(3)+1}{x+3} = \dfrac{5}{3}$

(ii) If $y \neq 1$, then substitute $k = 3$ in (3), $32y + 4 \geq 0$. $\quad \therefore \; y \geq -\dfrac{4}{32} = -\dfrac{1}{8}$

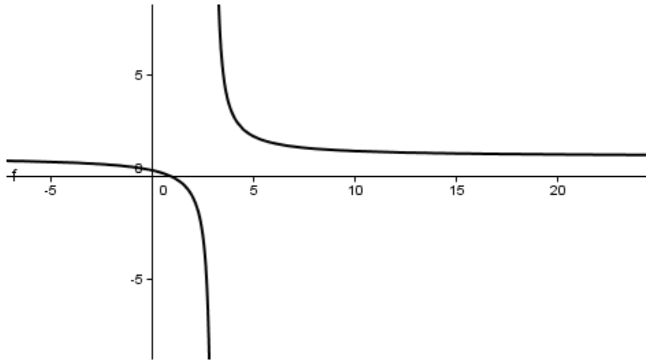
Hence the least integral value that y can take is 0 as it is the first integer $\geq -\dfrac{1}{8}$.

The graph with different values of k are shown below:

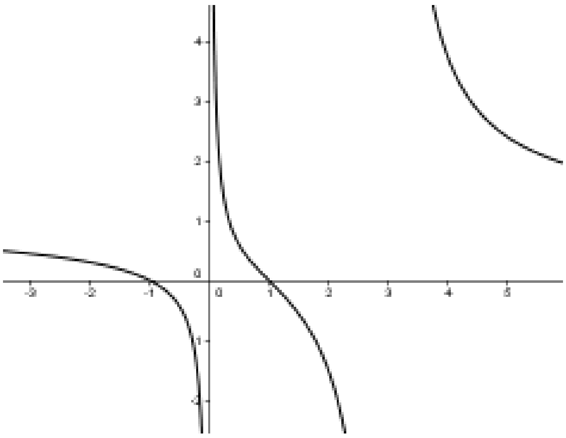
(1) k = –2



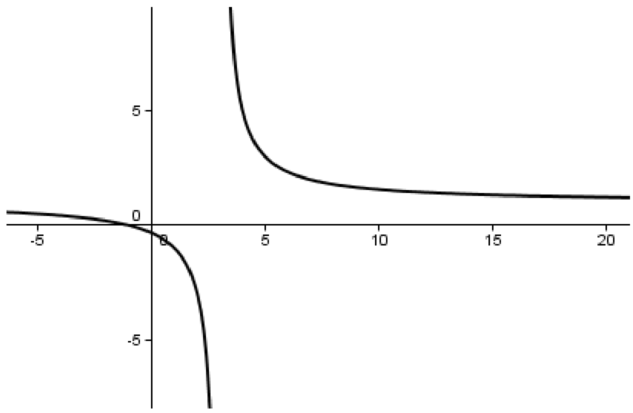
(2) k = –1



(3) k = 0



(4) k = 1



(5) k = 1.2